Root systems

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Chapter 1

Root pairings

1.1 Definitions

Recall that a *perfect pairing* is a quadruple $(R, M, N, \mathcal{L})$ where R is a commutative ring, M and N are R-modules, and $\mathcal{L} : M \times N \to R$ is a bilinear map such that the induced maps $M \to \mathrm{Hom}_R(N, R)$ and $N \to \mathrm{Hom}_R(M, R)$ are isomorphisms of R-modules. If I is a set, denote by $\mathrm{Perm}(I)$ the group of permutations of I.

Definition 1. A *root pairing* is a tuple $(R, M, N, \mathcal{L}, I, \alpha, \alpha^\vee, s)$ where:

- $(R, M, N, \mathcal{L})$ is a perfect pairing,
- I is a set,
- The maps

$$\begin{aligned} \alpha : I \to M, &\quad i \mapsto \alpha_i, \\ \alpha^\vee : I \to N, &\quad i \mapsto \alpha_i^\vee, \\ s : I \to \mathrm{Perm}(I), &\quad i \mapsto s_i, \end{aligned}$$

  are such that

  1. α and $\alpha^\vee$ are injective,
  2. for all $i \in I$, $\mathcal{L}(\alpha_i, \alpha_i^\vee) = 2$,
  3. for all $i, j \in I$,

$$\begin{aligned} \alpha_{s_i(j)} &= \alpha_j - \mathcal{L}(\alpha_j, \alpha_i^\vee)\alpha_i, \\ \alpha_{s_i(j)}^\vee &= \alpha_j^\vee - \mathcal{L}(\alpha_i, \alpha_j^\vee)\alpha_i^\vee. \end{aligned}$$

Definition 2. Fix a root pairing $\Phi = (R, M, N, \mathcal{L}, I, \alpha, \alpha^\vee, s)$. We say that Φ is:

- *crystallographic* if $\mathcal{L}(\alpha_i, \alpha_j^\vee) \in \mathbb{Z}$ for all $i, j \in I$,
- *reduced* if, for all $i, j \in I$, the roots α_i and α_j are linearly dependent over R if and only if $\alpha_i = \pm\alpha_j$,
- *finite* if I is finite.

Chapter 2

Constructions

2.1 Type A

Consider the perfect pairing

$$(R, M, N, \mathcal{L}) = (\mathbb{Z}, \mathbb{Z}^n, (\mathbb{Z}^n)^*, \langle \cdot, \cdot \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the canonical pairing between $\mathbb{Z}^n$ and its algebraic dual $(\mathbb{Z}^n)^* = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z})$. Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{Z}^n$, and let $e_1^*, \dots, e_n^*$ be the dual basis.

Write $[1, n] = \{1, \dots, n\}$. An *interval* in $[1, n]$ is a set $[i, j] = \{i, i+1, \dots, j\}$ with $1 \leq i \leq j \leq n$. We denote by I_n the set of intervals in $[1, n]$ and write $[i] = [i, i]$.

A *signed interval* in $[1, n]$ is an element of $I_n \times \{\pm 1\}$. We identify $J \in I_n$ with $(J, 1)$ and write $-J$ for $(J, -1)$.

Let $A = (a_{u,r})$ be the $n \times n$ integer matrix defined by

$$a_{u,r} = \begin{cases} 2 & \text{if } u = r, \\ -1 & \text{if } |u - r| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This is the Cartan matrix of type A_n.

Define

$$\alpha : I_n \times \{\pm 1\} \to \mathbb{Z}^n, \qquad \alpha_{\pm J} = \pm \sum_{r \in J} A e_r, \tag{2.1}$$

$$\alpha^\vee : I_n \times \{\pm 1\} \to (\mathbb{Z}^n)^*, \qquad \alpha^\vee_{\pm J} = \pm \sum_{r \in J} e_r^*. \tag{2.2}$$

Lemma 3. *Let $J = \varepsilon[i, j]$ and $K = \delta[k, l]$ be signed intervals. Then*

$$\mathcal{L}(\alpha_K, \alpha_J^\vee) = \varepsilon\delta\big(\delta_{i,k} + \delta_{j,l} - \delta_{i,l+1} - \delta_{k,j+1}\big).$$

Proof. It is enough to consider $J, K \in I_n$, since the signs factor out by bilinearity. For fixed r, the vector Ae_r has coefficient 2 in coordinate r, coefficient -1 in the adjacent coordinates $r-1$ and $r+1$, and coefficient 0 elsewhere. Hence, for $u \in [1, n]$,

$$\sum_{r=k}^{l} a_{u,r} = \delta_{u,k} + \delta_{u,l} - \delta_{u,k-1} - \delta_{u,l+1},$$

where $\delta_{x,y}$ is the Kronecker delta. Summing over $u = i, \dots, j$ gives the displayed formula. $\square$

Lemma 4. *For every $J \in I_n \times \{\pm 1\}$, $\mathcal{L}(\alpha_J, \alpha_J^\vee) = 2$.*

Proof. Write $J = \varepsilon[i, j]$. In Lemma 3, the two positive endpoint terms are both 1, while the two negative endpoint conditions $i = j + 1$ are impossible because $i \leq j$. Thus

$$\mathcal{L}(\alpha_J, \alpha_J^\vee) = \varepsilon^2(1 + 1) = 2.$$

$\square$

Lemma 5. *For signed intervals J and K,*

$$\mathcal{L}(\alpha_K, \alpha_J^\vee) = 2 \quad \Longleftrightarrow \quad K = J.$$

Proof. Write $J = \varepsilon[i, j]$ and $K = \delta[k, l]$. Let

$$E = \delta_{i,k} + \delta_{j,l} - \delta_{i,l+1} - \delta_{k,j+1}.$$

By Lemma 3, $\mathcal{L}(\alpha_K, \alpha_J^\vee) = \varepsilon\delta E$. The two negative endpoint conditions cannot both hold: if $i = l + 1$ and $k = j + 1$, then $l < i \leq j < k \leq l$, a contradiction. Hence $E \geq -1$. Since also $E \leq 2$, the equality $\varepsilon\delta E = 2$ forces $\varepsilon\delta = 1$ and $E = 2$.

The equality $E = 2$ forces both positive endpoint terms to be 1, so $i = k$ and $j = l$. Since $\varepsilon\delta = 1$ and $\varepsilon, \delta \in \{\pm 1\}$, we also have $\varepsilon = \delta$. Thus $K = J$. The converse is Lemma 4. $\square$

Lemma 6. *The maps $\alpha : I_n \times \{\pm 1\} \to \mathbb{Z}^n$ and $\alpha^\vee : I_n \times \{\pm 1\} \to (\mathbb{Z}^n)^*$ are injective.*

Proof. Suppose first that $\alpha_J = \alpha_K$. Pairing both sides with $\alpha_J^\vee$ gives

$$2 = \mathcal{L}(\alpha_J, \alpha_J^\vee) = \mathcal{L}(\alpha_K, \alpha_J^\vee).$$

Lemma 5 gives $K = J$.

Now suppose that $\alpha_J^\vee = \alpha_K^\vee$. Pairing both sides with α_J gives

$$2 = \mathcal{L}(\alpha_J, \alpha_J^\vee) = \mathcal{L}(\alpha_J, \alpha_K^\vee).$$

Applying Lemma 5 with the roles of the two signed intervals exchanged gives $J = K$. $\square$

We first define $s_J(K)$ for $J, K \in I_n$. Set $L = (J \cup K) \setminus (J \cap K)$. Exactly one of the following cases occurs:

R1. $L = \emptyset$.

R2. $L \in I_n$ and $K \subset J$.

R3. $L \in I_n$ and $K \not\subseteq J$.

R4. $L \notin I_n \cup \{\emptyset\}$.

Then

$$s_J(K) = \begin{cases} -K & \text{in case } \mathbf{R1}, \\ -L & \text{in case } \mathbf{R2}, \\ L & \text{in case } \mathbf{R3}, \\ K & \text{in case } \mathbf{R4}. \end{cases} \tag{2.3}$$

If $J=[i,j]$ and $K=[k,l]$, this is equivalently

$$s_{[i,j]}[k,l]=\begin{cases}-[k,l] & \text{if } (i,j)=(k,l),\\ -[l+1,j] & \text{if } i=k,\ j>l,\\ [j+1,l] & \text{if } i=k,\ j<l,\\ -[i,k-1] & \text{if } i<k,\ j=l,\\ [k,i-1] & \text{if } i>k,\ j=l,\\ [k,j] & \text{if } i=l+1,\\ [i,l] & \text{if } k=j+1,\\ [k,l] & \text{otherwise.}\end{cases} \tag{2.4}$$

Extend s_J to signed intervals by $s_J(-K)=-s_J(K)$, and extend the first argument by $s_{-J}=s_J$.

Lemma 7. *For all $J\in I_n\times\{\pm1\}$, we have $s_J\in\mathrm{Perm}(I_n\times\{\pm1\})$.*

Proof. A direct case check from (2.4) shows that $s_J(s_J(K))=K$ for every signed interval K. Hence s_J is its own inverse, and therefore a permutation. □

Lemma 8. *For all $J,K\in I_n\times\{\pm1\}$,*

$$\alpha_{s_J(K)}=\alpha_K-\mathcal{L}(\alpha_K,\alpha_J^\vee)\alpha_J.$$

Proof. First assume that $J=[i,j]$ and $K=[k,l]$ are positive intervals. Put $c=\mathcal{L}(\alpha_K,\alpha_J^\vee)$. Lemma 3 gives

$$c=\delta_{i,k}+\delta_{j,l}-\delta_{i,l+1}-\delta_{k,j+1}.$$

Comparing this value with (2.4) gives:

condition	c	$\alpha_K-c\alpha_J$
$(i,j)=(k,l)$	2	$-\alpha_{[k,l]}$
$i=k,\ j>l$	1	$-\alpha_{[l+1,j]}$
$i=k,\ j<l$	1	$\alpha_{[j+1,l]}$
$i<k,\ j=l$	1	$-\alpha_{[i,k-1]}$
$i>k,\ j=l$	1	$\alpha_{[k,i-1]}$
$i=l+1$	-1	$\alpha_{[k,j]}$
$k=j+1$	-1	$\alpha_{[i,l]}$
otherwise	0	$\alpha_{[k,l]}$.

Each entry in the last column is precisely $\alpha_{s_J(K)}$.

For arbitrary signs, write $J=\varepsilon J_0$ and $K=\delta K_0$ with $J_0,K_0\in I_n$ and $\varepsilon,\delta\in\{\pm1\}$. Since $s_{\varepsilon J_0}(\delta K_0)=\delta s_{J_0}(K_0)$ and

$$\mathcal{L}(\alpha_{\delta K_0},\alpha_{\varepsilon J_0}^\vee)=\delta\varepsilon\mathcal{L}(\alpha_{K_0},\alpha_{J_0}^\vee),$$

the positive case, multiplied by δ, proves the signed case. □

Lemma 9. *For all $J,K\in I_n\times\{\pm1\}$,*

$$\alpha_{s_J(K)}^\vee=\alpha_K^\vee-\mathcal{L}(\alpha_J,\alpha_K^\vee)\alpha_J^\vee.$$

Proof. For positive intervals, symmetry of the type A Cartan matrix gives the same endpoint value for $\mathcal{L}(\alpha_J, \alpha_K^\vee)$ as in the proof of Lemma 8. The case table there used only additivity of sums over adjacent or nested intervals. Replacing each Ae_r by e_r^* in that table gives the desired coroot identity for positive intervals. The signed case follows by the same sign reduction as in Lemma 8. □

Theorem 10. *Fix $n \in \mathbb{Z}_{>0}$, and let*

$$R = \mathbb{Z}, \qquad M = \mathbb{Z}^n, \qquad N = (\mathbb{Z}^n)^*.$$

Let $\mathcal{L}$ be the canonical pairing, let $I = I_n \times \{\pm 1\}$ be the set of signed intervals, let α and $\alpha^\vee$ be as in (2.1) and (2.2), and let s be as in (2.3). Then

$$(R, M, N, \mathcal{L}, I, \alpha, \alpha^\vee, s)$$

is a finite, crystallographic and reduced root pairing.

Proof. The quadruple $(\mathbb{Z}, \mathbb{Z}^n, (\mathbb{Z}^n)^*, \mathcal{L})$ is the standard perfect pairing between a finite free $\mathbb{Z}$-module and its dual. It remains to verify the root-pairing axioms from Definition 1.

The maps α and $\alpha^\vee$ are injective by Lemma 6. The identity $\mathcal{L}(\alpha_J, \alpha_J^\vee) = 2$ is Lemma 4. Lemma 7 gives $s_J \in \mathrm{Perm}(I)$ for each J. Finally, Lemmas 8 and 9 give the two reflection formulas. These are exactly the conditions in Definition 1. This proves the tuple is a root pairing. It is finite because I is finite, crystallographic by Lemma 3, and reduced by (2.1) and (2.2). □

2.2 Types B and C

We define a root pairing such that the roots form a type C root system. The set of coroots will be the type B root system.

- $R = \mathbb{Z}$,
- $M = \mathbb{Z}^n$,
- $N = (\mathbb{Z}^n)^*$,
- $\mathcal{L}$ is the canonical pairing,
- $I = \{(i, j, \varepsilon) : i, j \in \{1, \dots, n\}, \varepsilon \in \{\pm 1\}\}$,
- The map $\alpha : I \to M$ is given by

$$\alpha_{(i,j,\varepsilon)} = \begin{cases} \varepsilon(e_i + e_j), & \text{if } i \geq j, \\ \varepsilon(e_i - e_j), & \text{if } i < j. \end{cases}$$

- The map $\alpha^\vee : I \to N$ is given by

$$\alpha^\vee_{(i,j,\varepsilon)} = \begin{cases} \varepsilon(e_i^* + e_j^*), & \text{if } i > j, \\ \varepsilon e_i^*, & \text{if } i = j, \\ \varepsilon(e_i^* - e_j^*), & \text{if } i < j. \end{cases}$$

- The map $s : I \to \mathrm{Perm}(I)$ is given by...

2.3 Type D

2.4 Exceptional types

2.5 Non-reduced BC type

Chapter 3

Root pairings of finite type

Definition 11. Given a root pairing $\Phi = (R, M, N, \mathcal{L}, I, \alpha, \alpha^\vee, s)$, a set $\Delta \subset I$ is called a *base* of Φ if:

1. the sets $\{\alpha_i : i \in \Delta\}$ and $\{\alpha_i^\vee : i \in \Delta\}$ are R-linearly independent, and
2. for every $x \in I$, there exist $k, m \in \mathbb{Z}_{>0}$, $a_1, \dots, a_k, b_1, \dots, b_m \in \Delta$, and $\varepsilon, \delta \in \{\pm 1\}$ such that

$$\alpha_x = \varepsilon \sum_{r=1}^{k} \alpha_{a_r} \quad \text{and} \quad \alpha_x^\vee = \delta \sum_{r=1}^{m} \alpha_{b_r}^\vee.$$

Definition 12. A base Δ is said to be *irreducible* if there exist $k \in \mathbb{Z}_{>0}$ and $a_1, \dots, a_k \in \Delta$ such that:

1. $\{a_1, \dots, a_k\} = \Delta$, and
2. $\mathcal{L}(\alpha_{a_r}, \alpha_{a_{r+1}}^\vee) \neq 0$ for every $1 \leq r < k$.

By definition, a root pairing admitting an irreducible base is called *irreducible*.

Definition 13. Fix a finite, crystallographic, reduced and irreducible root pairing $\Phi = (R, M, N, \mathcal{L}, I, \alpha, \alpha^\vee, s)$ and a base $\Delta \subset I$. The pair (Φ, Δ) is said to be of:

1. type A if there is a sequence $(i_1, \dots, i_n)$, where $n = |\Delta|$, of distinct elements of Δ such that

$$\mathcal{L}(\alpha_{i_r}, \alpha_{i_{r+1}}^\vee) = -1 \quad \text{for all } 1 \leq r < n;$$

2. type B if there is a sequence $(i_1, \dots, i_n)$ of distinct elements of Δ such that

$$\mathcal{L}(\alpha_{i_r}, \alpha_{i_{r+1}}^\vee) = -1 \quad \text{for all } 1 \leq r < n-1, \qquad \mathcal{L}(\alpha_{i_n}, \alpha_{i_{n-1}}^\vee) = -2;$$

3. type C if there is a sequence $(i_1, \dots, i_n)$ of distinct elements of Δ such that

$$\mathcal{L}(\alpha_{i_r}, \alpha_{i_{r+1}}^\vee) = -1 \quad \text{for all } 1 \leq r < n-1, \qquad \mathcal{L}(\alpha_{i_{n-1}}, \alpha_{i_n}^\vee) = -2;$$

4. type D if there exist distinct elements $d_1, d_2, t \in \Delta$ such that

$$\mathcal{L}(\alpha_{d_a}, \alpha_t^\vee) = -1 \quad (a = 1, 2),$$

   and $\Delta \setminus \{d_1, d_2\}$ is of type A;

5. type E if there exist distinct elements $d_1, d_2, t \in \Delta$ such that

$$\mathcal{L}(\alpha_{d_1}, \alpha_{d_2}^\vee) = -1, \qquad \mathcal{L}(\alpha_{d_2}, \alpha_t^\vee) = -1,$$

and $\Delta \setminus \{d_1, d_2\}$ is of type A.

We now prove that the root pairings constructed in §2 exhaust all the root pairing types given in Definition 13.

Theorem 14. *Set $\Delta = \{[i] : i \in \{1, \dots, n\}\}$. The pair (Φ, Δ) where Φ is the root pairing constructed in Theorem 10 is a root pairing of type A.*

Proof. □