The Laurent Phenomenon

Antoine de Saint Germain

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Chapter 1

Matrix mutation

Definition 1.1. *1) An $n \times n$ matrix $B = (b_{i,j})$ with (say) rational entries is called* skew-symmetrisable *if there exists a diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_n)$ with $d_i \in \mathbb{Z}_{>0}$ such that DB is skew-symmetric, i.e.*

$$d_i b_{i,j} = -d_j b_{j,i}, \qquad \forall i, j \in [1, n].$$

2) A mutation matrix is a skew-symmetrisable matrix with integer entries.

Definition 1.2. *Let B be an $n \times n$ skew-symmetrisable matrix. The* mutation *of B in direction $k \in [1, n]$ is the matrix $\mu_k(B) = (b'_{i,j})$ defined by*

$$b'_{i,j} = \begin{cases} -b_{i,j} & \text{if } i = k \text{ or } j = k, \\ b_{i,j} + b_{i,k}b_{k,j} & \text{if } b_{i,k} > 0 \text{ and } b_{k,j} > 0, \\ b_{i,j} - b_{i,k}b_{k,j} & \text{if } b_{i,k} < 0 \text{ and } b_{k,j} < 0, \\ b_{i,j} & \text{otherwise.} \end{cases}$$

Lemma 1.3. *Let B be an $n \times n$ skew-symmetrisable matrix, and fix $k \in [1, n]$. Then the following hold:*

1. *$\mu_k(B)$ is skew-symmetrisable, with the same diagonal matrix D.*
2. *$\mu_k(\mu_k(B)) = B$.*
3. *If B is skew-symmetric, then $\mu_k(B)$ is skew-symmetric.*

Proof. 1. Fix $k \in [1, n]$, and let $i, j \in [1, n]$ be arbitrary. By definition of $\mu_k(B)$ and the skew-symmetry of B, we have

$$b'_{j,i} = \begin{cases} -b_{j,i} & \text{if } i = k \text{ or } j = k, \\ b_{j,i} - b_{j,k}b_{k,i} & \text{if } b_{i,k} > 0 \text{ and } b_{k,j} > 0, \\ b_{j,i} + b_{j,k}b_{k,i} & \text{if } b_{i,k} < 0 \text{ and } b_{k,j} < 0, \\ b_{j,i} & \text{otherwise.} \end{cases}$$

Suppose we are in the first case, i.e. $i = k$ or $j = k$. Then

$$d_i b'_{i,j} = d_i(-b_{i,j}) = -d_j(-b_{j,i}) = -d_j b'_{j,i}.$$

Now suppose we are in the second case, i.e. $b_{i,k} > 0$ and $b_{k,j} > 0$. Then

$$\begin{aligned} d_i b'_{i,j} &= d_i(b_{i,j} + b_{i,k}b_{k,j}) \\ &= -d_j b_{j,i} - d_k b_{k,i} b_{k,j} \\ &= -d_j b_{j,i} - d_k b_{k,i} b_{k,j} \\ &= -d_j b_{j,i} + d_j b_{k,i} b_{j,k} \\ &= -d_j(b_{j,i} - b_{j,k} b_{k,i}) \\ &= -d_j b'_{j,i}. \end{aligned}$$

The remaining two cases are similar, and we omit them.

2. This is a direct computation.

3. This follows from 1.

□