Coxeter frieze patterns

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Chapter 1

Field-valued patterns

Throughout this document, we will study frieze patterns of finite *height*. This terminology (as compared to finite *width* or finite *order*) is unconventional but more convenient for formalisation. For us, the height of a frieze pattern corresponds to the number of rows, including the rows of ones but *excluding* the rows of zeros. Throughout this section, we fix an arbitrary field F.

Definition 1.1. *Fix $n \in \mathbb{N}^*$. A map $f : \{0, 1, \dots, n, n+1\} \times \mathbb{Z} \longrightarrow F$ is called an F-valued pattern of height n if,*
1) for all $m \in \mathbb{Z}$, $f(0, m) = f(n+1, m) = 0$,
2) for all $m \in \mathbb{Z}$, $f(1, m) = f(n, m) = 1$, and
3) for all $(i, m) \in \{1, 2, \dots, n\} \times \mathbb{Z}$, we have

$$f(i, m)f(i, m+1) = 1 + f(i+1, m)f(i-1, m+1). \tag{1.1}$$

An F-valued pattern f of height n is said to be *nowhere zero* if $f(i, m) \neq 0$, for all $i \in \{1, \dots, n\}$ and for all $m \in \mathbb{Z}$.

Lemma 1.2. *Let f be a nowhere-zero F-valued pattern of height n. For all m, we have*

$$f(i+2, m) = f(2, m+i)f(i+1, m) - f(i, m), \qquad i \in \{0, \dots, n-1\}$$
$$f(i, m) = f(n-1, m)f(i+1, m-1) - f(i+2, m-2), \qquad i \in \{0, n-1\}.$$

Proof. We begin by proving the first statement. That is, we prove

$$P_i : \forall m \in \mathbb{Z}, f(i+2, m) = f(2, m+i+1)f(i+1, m) - f(i, m),$$

for $i \in \{0, \dots, n-1\}$. We do so by induction on i.

Base case P_0: We have that for all $m \in \mathbb{Z}$, $f(2, m)f(1, m) - f(0, m) = f(2, m+1) * 1 - 0 = f(2, m)$.

Inductive hypothesis. Suppose that our claim holds for some $i \in \{0, \dots, n-2\}$ fixed. Then,

$$\begin{aligned} f(i+3, m)f(i+1, m+1) &= f(i+2, m)f(i+2, m+1) - 1 \\ &= f(i+2, m)(f(2, m+i+1)f(i+1, m+1) - f(i, m+1)) - 1 \\ &= f(i+2, m)f(2, m+i+1)f(i+1, m+1) - (f(i+2, m)f(i, m+1) + 1) \\ &= f(i+2, m)f(2, m+i+1)f(i+1, m+1) - f(i+1, m)f(i+1, m+1). \end{aligned}$$

Since f is nowhere-zero, we may divide both sides of the equation by $f(i+1, m+1)$ to obtain the desired equality.

The second statement is proved almost identically. Namely, we prove

$$Q_i : \forall m \in \mathbb{Z}, f(i,m) = f(n-1,m)f(i+1,m-1) - f(i+2,m-2),$$

by induction on i, starting with $i = n-1$ and proving the inductive step $Q_i \Rightarrow Q_{i-1}$.

Base case Q_{n-1}: for all $m \in \mathbb{Z}$, $f(n-1,m)f(n,m-1) - f(n+1,m-2) = f(n-1,m) * 1 - 0 = f(n-1,m)$.

Inductive hypothesis. Suppose that Q_{i+1} holds for some fixed $i \in \{0, \dots, n-2\}$. Then,

$$\begin{aligned} f(i,m)f(i+2,m-1) &= f(i+1,m-1)f(i+1,m) - 1 \\ &= f(i+1,m-1)(f(i+2,m-1)f(n-1,m) - f(i+3,m-2)) - 1 \\ &= f(i+1,m-1)f(n-1,m)f(i+2,m-1) - (f(i+1,m-1)f(i+3,m-2) + 1) \\ &= f(i+1,m-1)f(n-1,m)f(i+2,m-1) - f(i+2,m-2)f(i+2,m-1). \end{aligned}$$

Again since f is nowhere-zero, dividing by $f(i+2,m-1)$ on both sides we obtain Q_i. □

Proposition 1.3. *Let f be a nowhere-zero F-valued pattern of height n. Then, for all $m \in \mathbb{Z}$ and all $i \in \{0, \dots, n+1\}$, we have*

$$f(i,m) = f(i,m+n+1).$$

Proof. We prove a stronger statement, called the *glide symmetry* of frieze patterns. First, consider the map $\rho_n : \{0,1,\dots,n+1\} \times \mathbb{Z} \longrightarrow \{0,1,\dots,n+1\} \times \mathbb{Z}$ given by

$$\rho_n(i,m) = (n+1-i, m+i). \tag{1.2}$$

We show that every nowhere-zero F-valued pattern of height n is ρ_n-invariant, i.e. satisfies

$$f(\rho_n(i,m)) = f(i,m), \qquad \forall (i,m) \in \{0,1,\dots,n+1\} \times \mathbb{Z}.$$

The proposition will then follow by observing that $\rho_n^2 : (i,m) \mapsto (i, m+n+1)$. Thus, consider the statement

$$P_i : \forall m \in \mathbb{Z}, f(i,m) = f(n+1-i, m+i),$$

where $i \in \{0, \dots, n+1\}$. To prove that P_i holds for all i, it is sufficient to prove that P_0, P_1 hold, and that $P_i \wedge P_{i+1} \Rightarrow P_{i+2}$.

P_0 : for all $m \in \mathbb{Z}$, $f(0,m) = 0 = f(n+1,m)$.

P_1 : for all $m \in \mathbb{Z}$, $f(1,m) = 1 = f(n,m+1)$.

Now suppose we are given $i \in \{0,1,\dots,n-1\}$ such that P_i and P_{i+1} hold. Then, for any fixed $m \in \mathbb{Z}$, we have

$$\begin{aligned} f(i+2,m) &= f(2,m+i)f(i+1,m) - f(i,m) \\ &= f(n-1,m+i+2)f(n-i,m+i+1) - f(n+1-i,m+i) \\ &= f(n-i-1,m+i+2). \end{aligned}$$

□

Corollary 1.4. *Let f be a nowhere-zero F-valued pattern of height n. Then,* $\mathrm{Im}(f) := \{f(i,m) : i \in \{1,\dots,n\}, m \in \mathbb{Z}\}$ *is a finite set.*

Proof. Consider the finite set $\mathcal{D} = \{(i,m) : i \in \{1,\dots,n\}, m \in \{0,\dots,n\}\}$. By Proposition 1.3,

$$\mathrm{Im}(f) = \{f(i,m) : (i,m) \in \mathcal{D}\},$$

and the right-hand side is obviously finite. □

Chapter 2

Pandean sequences, flutes and the Fibonacci sequence

Definition 2.1. *A sequence* (a_k)*, indexed by* $\mathbb{N}^*$ *and consisting of positive integers is called pandean if* $a_1 = 1$ *and if, for every* $k > 1$*, we have*

$$a_k \mid a_{k-1} + a_{k+1}.$$

Given a pandean sequence (a_k)*, if there exists a positive integer* n *such that* $a_k = a_{k+n-1}$ *for all* $k \in \mathbb{N}$*, the tuple* $(a_1, \dots, a_n)$ *is called a Pan flute, or simply a flute, of height* n*. The set of all flutes of a given height* n *is denoted* $Flute(n)$*.*

Note that in a flute of height n, the first and last entries are equal to 1.

Lemma 2.2. *For any positive integer* n*, the set* $Flute(n)$ *is non-empty.*

Proof. It is clear that the constant sequence consisting entirely of ones is pandean, and such a pandean sequence gives rise to a flute of height n for any n. □

Recall that the Fibonacci sequence $(F_k)_{k\in\mathbb{N}}$ is defined by $F_0 = 0, F_1 = 1$ and the recursive formula

$$F_k = F_{k-1} + F_{k-2}.$$

Lemma 2.3. *1) If* n *is odd, the* n*-tuple*

$$(F_2, F_4, F_6, \dots, F_{n-1}, F_n, F_{n-2}, F_{n-4}, \dots, F_5, F_3, F_1),$$

is a Pan flute of height n*.*

2) If n *is even, the* n*-tuple*

$$(F_2, F_4, F_6, \dots, F_{n-2}, F_n, F_{n-1}, F_{n-3}, \dots, F_5, F_3, F_1),$$

is a Pan flute of height n*.*

Proof. These are a tedious but straightforward calculation. □

Lemma 2.4. *In a flute* $(a_1, \dots, a_n)$*, one of the following two statements holds.*

1) $a_2 = 1$ *or* $a_{n-1} = 1$*.*

2) There exists an index $i \in \{2, \dots, n-1\}$ *such that* $a_i = a_{i-1} + a_{i+1}$*.*

Proof. Suppose that 1) does not hold. In particular, $a_2 - a_1 > 0$ and $a_n - a_{n-1} < 0$. We prove that statement 2) holds by contradiction. Thus, assume that for all $i \in \{2, \dots, n-1\}$, we have $a_i \neq a_{i-1} + a_{i+1}$. Since the a_i are positive integers, we necessarily have $a_{i-1} + a_{i+1} \geq 2a_i$ for all $i \in \{2, \dots, n-1\}$. Thus, for a given i, we have

$$a_{i+1} - a_i = a_{i+1} + a_{i-1} - a_{i-1} - a_i \geq 2a_i - a_{i-1} - a_i = a_i - a_{i-1}.$$

Gathering these inequalities, we obtain

$$a_n - a_{n-1} \geq a_{n-1} - a_{n-2} \geq \dots \geq a_2 - a_1 > 0,$$

which contradicts the fact that $a_n - a_{n-1} < 0$. □

Proposition 2.5. *Fix a positive integer n, and let $(a_1, \dots, a_n) \in \mathrm{Flute}(n)$. For any $i \in \{1, \dots, n\}$, we have $a_i \leq F_n$.*

Proof. Consider the statement

$$P_n : \text{If } (a_1, \dots, a_n) \in \mathrm{Flute}(n), \text{ then } a_i \leq F_n \text{ for all } i \in \{1, \dots, n\}.$$

The proposition claims that P_n holds for all $n \in \mathbb{N}^*$. We prove this by induction on n. The base case $n = 1$ is clear, since the only flute of height 1 is (1), and $F_1 = 1$. Similarly, the case $n = 2$ is clear, since the only flute of height 2 is $(1, 1)$, and $F_2 = 1$. Now, assume that P_k holds for all k up to and including a fixed $n \in \mathbb{N}^*$. Let $(a_1, \dots, a_{n+1}) \in \mathrm{Flute}(n+1)$. By Lemma 2.4, we have either $a_2 = 1$ or $a_n = 1$, or there exists an index $i \in \{2, \dots, n\}$ such that $a_i = a_{i-1} + a_{i+1}$. Suppose that $a_2 = 1$. Then $(a_1, a_3, \dots, a_{n+1}) \in \mathrm{Flute}(n)$, and so by the induction hypothesis, $a_i \leq F_n$ for all $i \in \{1, \dots, n\}$. Since $F_n \leq F_{n+1}$, we have $a_i \leq F_{n+1}$ for all $i \in \{1, \dots, n\}$. A similar argument applies if $a_n = 1$. Now, suppose that there exists an index $i \in \{2, \dots, n\}$ such that

$$a_i = a_{i-1} + a_{i+1}. \tag{2.1}$$

We claim that $(a_1, a_2, \dots, a_{i-1}, \widehat{a_i}, a_{i+1}, \dots, a_{n+1}) \in \mathrm{Flute}(n)$, where $\widehat{a_i}$ means we omit a_i. Again by the induction hypothesis, we have that

$$a_j \leq F_n, \qquad j \neq i. \tag{2.2}$$

It remains to show that $a_i \leq F_{n+1}$. To see this, note that (2.1) and (2.2) together imply it is sufficient to show that either a_{i-1} or a_{i+1} is bounded above by F_{n-1}. But recall that a_{i-1} and a_{i+1} both belong to the flute $(a_1, a_2, \dots, a_{i-1}, \widehat{a_i}, a_{i+1}, \dots, a_{n+1})$ of height n. Thus, the conditions of Lemma 2.4 apply to this flute, so that by a reduction argument identical to the one above, there exists a flute of height $n-1$ containing either a_{i-1} or a_{i+1} (or both !), whereby at least one of the two is bounded above by F_{n-1}. This completes the induction step, and the proposition follows. □

Chapter 3

Maximal values of arithmetic frieze patterns

Definition 3.1. *A $\mathbb{Q}$-valued pattern of height n is said to be an arithmetic frieze pattern if it takes values in $\mathbb{Z}_{>0}$. We denote by Frieze(n) the set of arithmetic frieze patterns of height n.*

The following proposition is a key result connecting arithmetic frieze patterns to flutes.

Proposition 3.2. *1) Let f be an arithmetic frieze pattern of height n. For all $m \in \mathbb{Z}$, the n-tuple*

$$(f(1,m), f(2,m), \dots, f(n,m))$$

is a flute of height n.

2) Given a flute $(a_1, \dots, a_n)$, there exists a arithmetic frieze pattern f of height n such that

$$(f(1,0), \dots, f(n,0)) = (a_1, \dots, a_n).$$

Proof. 1) Note that we have $f(1,0) = f(n,0) = 1$ by definition. Moreover, f is arithmetic and so the first equation in Lemma 1.2 is precisely the divisibility condition defining a flute.

2) By arguing recursively, one can construct a pattern f such that $(f(1,0), \dots, f(n,0)) = (a_1, \dots, a_n)$. Moreover, such a frieze pattern is necessarily positive and $\mathbb{Q}$-valued. It remains to show that f is integer-valued. We begin by showing that $f(2,m) \in \mathbb{Z}$ for all $m \in \mathbb{Z}$. By the definition of a flute, $f(2,0) \in \mathbb{Z}$, and for each $i \in \{1, \dots, n-2\}$, there exists a positive integer c_i such that

$$f(i+1,0) * c_i = f(i+2,0) + f(i,0).$$

Using the first equation in Lemma 1.2, we deduce that $f(2,i) \in \mathbb{Z}$ for $i = 0, \dots, n-2$. Moreover, $f(2,n-1) = f(n-1,0) \in \mathbb{Z}$ by assumption. Thus we have proved that $f(2,m) \in \mathbb{Z}$ for $m = 0, \dots, n-1$. To see that $f(2,n) \in \mathbb{Z}$, note from Lemma 1.2 that $f(2,n) = f(n-1,1)$ can be expressed as a *polynomial* with integer coefficients in the variables $f(2,1), f(2,2), \dots, f(2,n-2)$. The claim for all m then follows from Proposition 1.3.

To see how this implies that $f(i,m) \in \mathbb{Z}$ for all $i \in \{2, \dots, n\}$, it suffices to see, again from Lemma 1.2, that every $f(i,m)$ can be expressed as a *polynomial* with integer coefficients in the variables $f(2,m), f(2,m+1), \dots, f(2,m+i-2)$.

□

Corollary 3.3. *Fix a positive integer n. The set Frieze(n) is non-empty.*

Proof. The proof of Lemma 2.2 showed that $(1, 1, \dots, 1)$ is a flute. The claim then follows from 2) of Proposition 3.2. ☐

By combining Proposition 3.2 with Proposition 2.5, we see that for each n, there is a well-defined positive integer, called the *maximum value* among arithmetic frieze patterns of height n, defined by

$$u_n := \max(f(i, m) : f \in \text{Frieze}(n), i \in \{1, \dots, n\}, m \in \mathbb{Z}).$$

We are now able to formulate and prove the main theorem of this section.

Theorem 3.4. *For all $n \geq 1$, we have*

$$u_n = F_n.$$

Proof. By Proposition 3.2, every entry of an arithmetic frieze pattern of height n belongs to a flute of height n. By Proposition 2.5, entries in a flute of height n are bounded above by F_n. Thus $u_n \leq F_n$ for all n. On the other hand, Lemma 2.3 and 2) of Proposition 3.2 show that there exists an arithmetic frieze pattern of height n containing F_n as a value. ☐